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Spline Approximation Operators of Bernstein–Schoenberg Type in One and Two Variables

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1. INTRODUCTION

In [13], Schoenberg introduced the spline operators

$$V_k(f; \cdot) = \sum_{i=1}^{n} f(\xi_i)\, N(\cdot \mid t_i, \ldots, t_{i+k}) \tag{1.1}$$

which reproduce linear functions and are variation-diminishing. They also have the shape-preserving properties of Bernstein polynomials to which they reduce with appropriate choice of knots (t_i). The approximation properties of these operators were further investigated by Marsden and Schoenberg [9] and Marsden [10].

More recently C. de Boor [1] highlighted the geometric interpretation of B-splines due to Curry and Schoenberg [3] and extended this to give a definition of B-splines in higher dimensions. Subsequently C. A. Micchelli [11] and W. Dahmen [5] obtained some analytic properties of these B-splines together with some recurrence relations. In [4] Dahmen constructed a class of these B-splines whose linear span contains all polynomials of appropriate degree.

In this paper we shall use the geometric definition of B-splines to construct spline approximation operators of type (1.1) in one and two dimensions. In the case of one dimension we allow different orderings of the knots for the B-splines in (1.1). The main tool will be a generalisation of an identity of

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Marsden [10] which we prove in one dimension in Section 2 and in two dimensions in Section 5. In [4] Dahmen has given a different generalisation of Marsden's identity and our proof is similar to his, both using elementary geometrical methods. Our identities differ from Dahmen's in using a triangulation of a simplex rather than a cube, and producing a single identity involving 2 parameters (in 2 dimensions) rather than a class of identities involving a single parameter. These allow us, in the one- and two-dimensional cases considered, to find simple, explicit formulas for the B-spline coefficients in the identities.

In Section 3 we study the variation-diminishing property of the operators in one dimension, while in Sections 4 and 6 we prove convergence results for the operators in one and two dimensions, respectively. These operators reduce to Bernstein polynomials when restricted to a triangular domain with appropriate choice of triangulation, as is shown in Section 7. An important feature of these spline operators is that they are defined on any polygonal domain in $\mathbb{R}^2$ and not restricted to triangles or squares.

2. A Geometric Proof of a Generalised Marsden's Identity

We first introduce the B-splines defined in [1]. For $0 < s \leqslant k$, and δ a k-simplex in $\mathbb{R}^k$, we define

$$M_\delta(x) := \mathrm{vol}_{k-s}(\{v \in \delta : pv = x\}), \qquad \forall x \in \mathbb{R}^s, \tag{2.1}$$

where vol_d means d-dimensional volume, and $p: \mathbb{R}^k \to \mathbb{R}^s$ denotes the projection $pv := (v_i)_{i=1}^s$. Then M_δ is a piecewise polynomial function of degree $\leqslant k-s$ with compact support in $\mathbb{R}^s$. If δ has vertices $v^0,\dots, v^k$ and $x^i = pv^i$, then it is known [5, 11] that $M_\delta(x)/\mathrm{vol}_k \delta$ depends only on $x^0,\dots, x^k$ and so we can define the B-spline

$$M(x \mid x^0,\dots, x^k) := M_\delta(x)/\mathrm{vol}_k \delta, \qquad \forall x \in \mathbb{R}^s. \tag{2.2}$$

Now take $k > 1$ and let Δ_{k-1} denote the standard $(k-1)$-simplex in $\mathbb{R}^{k-1}$ with vertices the standard vectors

$$e^i := (\delta_{ij})_{j=1}^{k-1}, \qquad i = 0,\dots, k-1. \tag{2.3}$$

Let $\Delta = \{\delta_i\}_{i=1}^n$ be a triangulation of $[0, 1] \times \Delta_{k-1} \subset \mathbb{R}^k$ such that each vertex lies on one of the edges $\mathscr{F}_j = \{(x, e^j): x \in [0, 1]\}$, $j = 0,\dots, k-1$. For $i = 1,\dots, n$, we denote the vertices of δ_i by $v^{i0},\dots, v^{ik}$ and their projection on the x_1-axis by $t_{i0},\dots, t_{ik}$. Note that for any simplex δ_i, each edge $\mathscr{F}_j$, $j = 0,\dots, k-1$, contains at least one of the vertices of δ_i and hence there is exactly one such edge containing exactly two vertices of δ_i. We shall denote these two vertices by v^{i0} and v^{ik}, where $t_{i0} < t_{ik}$.

Now for $i = 1, \ldots, n$ we define a normalised B-spline of degree $k-1$ on $[0, 1]$ by

$$N_i := (k-1)!\, M_{\delta_i}. \tag{2.4}$$

We wish to calculate $\text{vol}_k \delta_i$. If v^{i0} and v^{ik} lie on $\mathscr{F}_1$, then $\text{vol}_k \delta_i$ is the absolute value of

$$\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_1^{i0} & v_1^{i1} & \cdots & v_1^{ik} \\ \vdots & \vdots & & \vdots \\ v_k^{i0} & v_k^{i1} & \cdots & v_k^{ik} \end{vmatrix}$$

which, after reordering $v^{i1}, \ldots, v^{i(k-1)}$ if necessary, equals the absolute value of

$$\frac{1}{k!}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ t_{i0} & t_{ik} & t_{i1} & t_{i2} & t_{i3} & \cdots & t_{i(k-1)} \\ 1 & 1 & 0 & 0 & 0 & & \\ & & & 1 & 0 & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{vmatrix}.$$

Thus

$$\text{vol}_k\, \delta_i = \frac{1}{k!}(t_{ik} - t_{i0}). \tag{2.5}$$

A similar calculation shows that (2.5) also holds when v^{i0}, v^{ik} lie on $\mathscr{F}_j$; $j \neq 1$. Then by (2.2), (2.4), (2.5),

$$N_i = \frac{1}{k}(t_{ik} - t_{i0})\, M(\cdot \mid t_{i0}, \ldots, t_{ik}). \tag{2.6}$$

Thus N_i depends only on $t_{i0}, \ldots, t_{ik}$, and we may write

$$N_i = N(\cdot \mid t_{i0}, \ldots, t_{ik}).$$

We have the following generalisation of Marsden's identity.

THEOREM 2.1. *For any* $y \in \mathbb{R}$ *and* $0 \leqslant x \leqslant 1$,

$$(y-x)^{k-1} = \sum_{i=1}^{n} \left\{ \prod_{j=1}^{k-1} (y-t_{ij}) \right\} N_i(x). \tag{2.7}$$

Proof. Take any $y > 1$ and let Δ_k^y denote the k-simplex in $\mathbb{R}^k$ with vertices (y, e^0), $(0, e^i)$, $i = 0,..., k-1$. Let $P := \{(x_1,..., x_k) \in \Delta_k^y : 0 \leqslant x_1 \leqslant 1\}$. A simple geometric argument shows $P = \{T_y v : v \in [0, 1] \times \Delta_{k-1}\}$, where for $(x, z) \in [0, 1] \times \Delta_{k-1}$, $T_y(x, z) = (x, (y-x)z/y)$. For $i = 1,..., n$, let δ_i^y denote the simplex $[T_y v^{i0},..., T_y v^{ik}]$, where $[T]$ denotes the convex hull of the set T. Then $\Delta^y := \{\delta_i^y\}_{i=1}^n$ is a triangulation of P. By a calculation similar to that for (2.5),

$$\text{vol}_k \delta_i^y = \frac{1}{k!} (t_{ik} - t_{i0}) \prod_{j=1}^{k-1} \left(\frac{y - t_{ij}}{y} \right). \tag{2.8}$$

Now for each x in $[0, 1]$, the hyperplane $\{(x_1,..., x_k) : x_1 = x\}$ intersects P in the $(k-1)$-simplex σ_x with vertices $T_y(x, e^j)$, $j = 0,..., k-1$. Then

$$\begin{aligned}
\frac{1}{(k-1)!} \left(\frac{y-x}{y} \right)^{k-1} &= \text{vol}_{k-1} \sigma_x \\
&= \sum_{i=1}^{n} M_{\delta_i^y}(x) \qquad \text{by} \quad (2.1) \\
&= \sum_{i=1}^{n} \text{vol}_k \delta_i^y M(x \mid t_{i0},..., t_{ik}) \qquad \text{by} \quad (2.2) \\
&= \sum_{i=1}^{n} \frac{1}{(k-1)!} \left\{ \prod_{j=1}^{k-1} \left(\frac{y-t_{ij}}{y} \right) \right\} N_i(x) \qquad \text{by} \quad (2.6), (2.8)
\end{aligned}$$

which gives (2.7). ∎

COROLLARY. *For* $k > 2$ *and* $i = 1,..., n$ *let*

$$\xi_i = \frac{1}{k-1} \sum_{j=1}^{k-1} t_{ij}, \tag{2.9}$$

$$\xi_i^{(2)} = \frac{2}{(k-1)(k-2)} \sum_{0<l<m<n} t_{il} t_{im}. \tag{2.10}$$

Then for all x *in* $[0, 1]$,

$$1 = \sum_{i=1}^{n} N_i(x), \tag{2.11}$$

$$x = \sum_{i=1}^{n} \xi_i N_i(x), \tag{2.12}$$

$$x^2 = \sum_{i=1}^{n} \xi_i^{(2)} N_i(x). \tag{2.13}$$

Proof. In (2.7) equate coefficients of $y^{k-1}, y^{k-2}, y^{k-3}$, respectively.

Remark. Suppose $0 = t_1 = \cdots = t_k < t_{k+1} \leqslant \cdots \leqslant t_n < t_{n+1} = \cdots = t_{n+k} = 1$, and for $i = 1,\dots, n$, $t_i = \cdots = t_{i+j} \Rightarrow j < k$. Then the following triangulation Δ gives rise to the usual B-splines of Curry and Schoenberg [3] and the usual Marsden's identity [10]:

$$\delta_1 = [(t_1, e^0), (t_2, e^1),\dots, (t_k, e^{k-1}), (t_{1+k}, e^0)],$$

$$\delta_2 = [(t_2, e^1), (t_3, e^2),\dots, (t_{1+k}, e^0), (t_{2+k}, e^1)],$$

$$\vdots$$

$$\delta_n = [(t_n, e^l), (t_{n+1}, e^{l+1}),\dots, (t_{n+k}, e^l)],$$

where $l \equiv n-1 \pmod k$.

We shall refer to this triangulation as the *usual triangulation*. In this case the normalised B-splines are

$$N_i = N(\cdot \mid t_i,\dots, t_{i+k}), \qquad i = 1,\dots, n.$$

3. The Variation-Diminishing Property of Bernstein–Schoenberg Operators

For $k > 1$ define a triangulation $\Delta = \{\delta_i\}_{i=1}^{n}$ as in Section 2 and for $i = 1,\dots, n$, define N_i, ξ_i as in (2.6), (2.9). Then we define the following spline operator V_Δ, which we call a Bernstein–Schoenberg operator.

For any function f on $[0, 1]$,

$$V_\Delta(f; x) := \sum_{i=1}^{n} f(\xi_i)\, N_i(x), \qquad \forall x \in [0, 1]. \tag{3.1}$$

If Δ is the usual triangulation, V_Δ reduces to the operator V_k of Schoenberg (1.1). Clearly V_Δ is a positive linear operator. It follows from (2.11) and (2.12) that V_Δ reproduces polynomials of degree 1.

An important property of the operators V_k is that they are variation-diminishing, i.e., for any f,

$$S^-(V_k(f; \cdot)) \leqslant S^-(f). \tag{3.2}$$

where S^- denotes the number of strong sign changes in $[0, 1]$, see [2, 13].

For a general triangulation Δ, the operator V_Δ need not be variation-diminishing as the following example shows. Let $k = 2$ and define $\Delta = \{\delta_i\}_{i=1}^4$ by

$$\delta_1 = [(0,0), (0,1), (\tfrac{5}{8},0)],$$

$$\delta_2 = [(0,1), (\tfrac{5}{8},0), (\tfrac{3}{8},1)],$$

$$\delta_3 = [(\tfrac{5}{8},0), (\tfrac{3}{8},1), (1,0)],$$

$$\delta_4 = [(\tfrac{3}{8},1), (1,0), (1,1)].$$

If f is any function with $S^-(f) = 1$, $f(0) = f(\frac{3}{8}) = 1$, $f(\frac{5}{8}) = f(1) = -1$, then it is easily seen that $S^-(V_\Delta(f)) = 3$.

However we do have the following result.

THEOREM 3.1. *Let $\Delta = \{\delta_i\}_{i=1}^n$ be a triangulation of $[0,1] \times \Delta_{k-1}$ satisfying*

$$t_{ij} \leqslant t_{ik}, \qquad j = 0,\dots,k, \qquad i = 1,\dots,n, \tag{3.3}$$

or

$$t_{i0} \leqslant t_{ij}, \qquad j = 0,\dots,k, \qquad i = 1,\dots,n. \tag{3.4}$$

Then V_Δ is variation-diminishing.

We note that this includes the case V_k because Δ is the usual triangulation if and only if

$$t_{i0} \leqslant t_{ij} \leqslant t_{ik}, \qquad j = 0,\dots,k, \qquad i = 1,\dots,n.$$

The proof of Theorem 3.1 requires some subsidiary results and first we order the simplices of Δ as follows (without yet imposing conditions (3.3) or (3.4)). Let δ_1 be the unique simplex containing the $(k-1)$-face $\{0\} \times \Delta_{k-1}$. There is then a unique simplex having a common $(k-1)$-face with δ_1 and this simplex is denoted by δ_2. Continuing in this manner, suppose for some $1 < i < n$ we have defined $\delta_1,\dots,\delta_i$ so that for $j = 1,\dots,i-1$, δ_j and δ_{j+1} have a common $(k-1)$-face. Then there are precisely two simplices having a common $(k-1)$-face with δ_i. One of these is δ_{i-1} and the other we denote by δ_{i+1}. This gives a unique ordering of Δ and δ_n is the unique simplex containing the $(k-1)$-face $\{1\} \times \Delta_{k-1}$.

Throughout the rest of this section we assume Δ has this ordering. We note that for $i = 2,\dots,n$, δ_i contains precisely one vertex, namely, v_{ik}, not contained in any δ_j, $j < i$. Thus the total number of vertices is $(k+1) + (n-1) = n+k$. We denote the projections of these vertices on $[0,1]$ by $\mathbf{t} = \{t_i\}_{i=1}^{n+k}$, where $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n+k}$. Clearly $t_1 = \cdots = t_k = 0$, $t_{n+1} = \cdots = t_{n+k} = 1$ and if for any i, $t_i = \cdots = t_{i+j}$, then $j < k$. The usual B-

splines $M(\cdot \mid t_i, \ldots, t_{i+k})$, $i = 1, \ldots, n$, thus form a basis for the space $\mathcal{S}(\mathbf{t})$ of all splines on $[0, 1]$ with knots in $\mathbf{t}$. Hence the B-splines $N_1, \ldots, N_n$ form a basis for $\mathcal{S}(\mathbf{t})$ if and only if they are linearly independently. They are not linearly independent in general as is seen by the following example.

Let $k = 2$ and define $\Delta = \{\delta_i\}_{i=1}^4$ by

$$\delta_1 = [(0,0), (0,1), (\tfrac{1}{2}, 0)],$$

$$\delta_2 = [(\tfrac{1}{2}, 0), (0,1), (1,0)],$$

$$\delta_3 = [(0,1), (1,0), (\tfrac{1}{2}, 1)],$$

$$\delta_4 = [(\tfrac{1}{2}, 1), (1,0), (1,1)].$$

Then, clearly $N_2 = N_3$.

However, we do have the following result.

PROPOSITION 3.1. *Suppose the triangulation Δ is such that all vertices projected onto the same point on* $[0, 1]$ *lie in a common simplex in Δ. Then $N_1, \ldots, N_n$ are linearly independent.*

Proof. For $i = 1, \ldots, n$ we shall prove by induction that $N_1, \ldots, N_i$ are linearly independent. Suppose then that for some $1 < i \leqslant n$, $N_1, \ldots, N_{i-1}$ are linearly independent. Let S denote the set of vertices of $\delta_1, \ldots, \delta_i$ whose projections on $[0, 1]$ equal t_{ik}. By assumption S comprises vertices of a common simplex, say, δ_j. We cannot have $j < i$ since $v_{ik} \in S$ and $v_{ik} \notin \delta_j$ for $j < i$. But if any element of S lies in δ_j for $j > i$, then it also lies in δ_i, by the nature of the ordering of Δ. Thus S must comprise vertices of δ_i.

Let r denote the cardinality of S and suppose that for some numbers $\lambda_1, \ldots, \lambda_i$, $f := \lambda_1 N_1 + \cdots \lambda_i N_i = 0$. Then $0 = f^{(k-r)}(t_{ik}^+) - f^{(k-r)}(t_{ik}^-) = \lambda_i \{N_i^{(k-r)}(t_{ik}^+) - N_i^{(k-r)}(t_{ik}^-)\}$ and hence $\lambda_i = 0$. So $\lambda_1 N_1 + \cdots + \lambda_{i-1} N_{i-1} = 0$ and since $N_1, \ldots, N_{i-1}$ are linearly independent, $\lambda_1 = \cdots = \lambda_{i-1} = 0$. Thus $N_1, \ldots, N_i$ are linearly independent and the inductive step is complete. ∎

COROLLARY. *If Δ satisfies* (3.3) *or* (3.4), *then $N_1, \ldots, N_n$ are linearly independent.*

Proof. First note that for $i = 1, \ldots, n-1$ the $(k-1)$-face in common to δ_i and δ_{i+1} is given by $\{v_{ij} : j = 1, \ldots, k\}$ and $\{v_{i+1,j} : j = 0, \ldots, k-1\}$. We now suppose Δ satisfies (3.3) (the case (3.4) following similarly) and note that (3.3) is equivalent to the condition $t_{1k} \leqslant t_{2k} \leqslant \cdots \leqslant t_{nk}$.

Let S denote a set of vertices having the same projection on $[0, 1]$. We have to show the elements of S lie in a common simplex. If the projection is 0, then clearly $S \subset \delta_1$. Otherwise S is of the form $\{v_{ik} : \alpha \leqslant i \leqslant \beta\}$ for some $1 \leqslant \alpha \leqslant \beta \leqslant n$. We shall show by induction on j that for $\alpha \leqslant j \leqslant \beta$, $\{v_{ik} : \alpha \leqslant i \leqslant j\} \subset \delta_j$. This is trivially true for $j = \alpha$. Suppose it is true for some j,

$\alpha \leqslant j < \beta$. Then every element of $\{v_{ik}: \alpha \leqslant i \leqslant j\}$ is of the form v_{jl} for some l, and since the projection of v_{jl} on $[0, 1]$ is $t_{jk} > t_{j0}$, we must have $l > 0$, and hence $v_{jl} \in \delta_{j+1}$. So $\{v_{ik}: \alpha \leqslant i \leqslant j \leqslant j+1\} \subset \delta_{j+1}$ and by induction $S = \{v_{ik}: \alpha \leqslant i \leqslant \beta\} \subset \delta_\beta$. ∎

Henceforth in this section we assume Δ satisfies (3.3) or (3.4). We will prove the results for case (3.3), case (3.4) following similarly.

LEMMA 3.1. *For any p, q with $1 \leqslant p \leqslant p+q-1 \leqslant n$, and for any points $\tau_1 < \cdots < \tau_q$ in $[0, 1]$, $\det(N_{p-1+i}(\tau_j))_{i,j=1}^q \geqslant 0$.*

Proof. The proof is by induction on q and follows the ideas of de Boor [2]. Clearly the result is true for $q = 1$. Take $1 < r \leqslant n$ and suppose it is true for $q = r - 1$. Now choose any p with $1 \leqslant p \leqslant p+r-1 \leqslant n$ and points $\tau_1 < \cdots < \tau_r$ in $[0, 1]$. We must show $\det(N_{p-1+i}(\tau_j))_{i,j=1}^r \geqslant 0$.

By an earlier argument we know $\delta_p, \ldots, \delta_{p+r-1}$ have a total of $r + k$ distinct vertices. Denote the projections of these vertices on $[0, 1]$ by $\mathbf{s} = \{s_i\}_1^{r+k}$, where $s_1 \leqslant \cdots \leqslant s_{r+k}$, and let $\mathscr{S}(\mathbf{s})$ denote the space spanned by $M_i := M(\cdot \mid s_i, \ldots, s_{i+k})$, $i = 1, \ldots, r$. By the corollary to Proposition 3.1, $N_p, \ldots, N_{p+r-1}$ are linearly independent and so form a basis for $\mathscr{S}(\mathbf{s})$. It is well-known (see [1]) that $\det(M_i(\tau_j))_{i,j=1}^r \neq 0$ iff $s_i < \tau_i < s_{i+k}$, $i = 1, \ldots, r$. We may assume $\det(N_{p-1+i}(\tau_j))_{i,j=1}^r \neq 0$ and hence that $s_i < \tau_i < s_{i+k}$, $i = 1, \ldots, r$. It follows that $\det(M_i(\tau_j))_{i,j=1}^{r-1} \neq 0$ and since $N_p, \ldots, N_{p+r-2}$ span the same space as $M_1, \ldots, M_{r-1}$, we have by the induction hypothesis, $\det(N_{p-1+i}(\tau_j))_{i,j=1}^{r-1} > 0$.

Now for x in $[0, 1]$ define $f(x) = \det(N_{p-1+i}(\hat{\tau}_j))_{i,j=1}^r$, where $\hat{\tau}_j = \tau_j$, $j = 1, \ldots, r-1$, and $\hat{\tau}_r = x$. Then f is a linear combination of $N_p, \ldots, N_{p+r-1}$. If s_{r+k} has multiplicity α in $\mathbf{s}$, then $N_{p+r-1}^{(k-\alpha)}(s_{r+k}^-) \neq 0$ and $N_i^{(k-\alpha)}(s_{r+k}^-) = 0$ for $p \leqslant i < p+r-1$. Thus in $(s_{r+k} - \varepsilon, s_{r+k})$ for small enough $\varepsilon > 0$, f is dominated by the term involving N_{p+r-1} and so for x in $(s_{r+k} - \varepsilon, s_{r+k})$, $f(x)$ has the same sign as the coefficient of N_{p+r-1}, namely, $\det(N_{p-1+i}(\tau_j))_{i,j=1}^{r-1} > 0$. But $f(x)$ cannot vanish or change sign for x in $[\tau_r, s_{r+k})$ and so $f(\tau_r) > 0$, i.e., $\det(N_{p-1+i}(\tau_j))_{i,j=1}^r > 0$. ∎

LEMMA 3.2. *For any points $\tau_1 < \cdots < \tau_n$ in $[0, 1]$, the matrix $\|N_i(\tau_j)\|_{i,j=1}^n$ is totally positive.*

Proof. This follows from Lemma 3.1 by applying the method of Karlin [6, p. 528].

Proof of Theorem 3.1. It follows easily from (3.3) or (3.4) that $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_n$. But it is known [6] that for any totally positive matrix A and any vector x (of appropriate length), the vector Ax has no more sign changes than does x. The result then follows immediately from Lemma 3.2. ∎

4. Convergence of Bernstein–Schoenberg Operators

In this section we assume $k > 2$. In [10] Marsden showed that $V_k(f; \cdot) \to f$ uniformly on $[0, 1]$ for all f in $C[0, 1]$ if $k^{-1}(\max \Delta t_i) \to 0$. We now generalise this result to the operators V_Δ.

THEOREM 4.1. *For every f in $C[0, 1]$, $V_\Delta(f; \cdot) \to f$ uniformly on $[0, 1]$ if $(1/k) \max_{1 \leqslant i \leqslant n} d_i \to 0$, where $d_i = \max_j t_{ij} - \min_j t_{ij}$.*

Proof. Define g in $C[0, 1]$ by $g(x) = x^2$. Since V_Δ is a positive linear operator which reproduces polynomials of degree 1, it follows from the Bohman–Korovkin Theorem (see [7]) that $V_\Delta(f; \cdot) \to f$ uniformly on $[0, 1]$ for all f in $C[0, 1]$ if $V_\Delta(g; \cdot) \to g$ uniformly on $[0, 1]$.

Now from (3.1) and (2.13) we have

$$V_\Delta(g; x) - g(x) = \sum_{i=1}^{n} (\xi_i^2 - \xi_i^{(2)}) N_i(x). \tag{4.1}$$

A straightforward calculation shows that for $i = 1, \ldots, n$,

$$\xi_i^2 - \xi_i^{(2)} = \frac{1}{(k-1)^2 (k-2)} \sum_{0 < l < m < k} (t_{im} - t_{il})^2$$

and thus $0 \leqslant \xi_i^2 - \xi_i^{(2)} \leqslant d_i^2 / 2(k-1)$.

So by (4.1) and (2.11) we have

$$0 \leqslant V_\Delta(g; x) - g(x) \leqslant \frac{1}{2(k-1)} \max_{1 \leqslant i \leqslant n} d_i^2, \qquad \forall x \in [0, 1].$$

Hence $V_\Delta(g; \cdot) \to g$ uniformly on $[0, 1]$ if $1/k \max_{1 \leqslant i \leqslant n} d_i \to 0$. ∎

Defining $t = \{t_i\}_{i=1}^{n+k}$ as in Section 3, the conclusion of Theorem 4.1 need not be valid if we merely assume $k^{-1}(\max_i \Delta t_i) \to 0$. As a counterexample let $k = 3$ and take any points $0 = t_1 = t_2 = t_3 < t_4 < \cdots < t_{n+1} = t_{n+2} = t_{n+3} = 1$. Define $\Delta = \{\delta_i\}_{i=1}^n$ by

$$\delta_i = [(t_{i+2}, 0, 0), (t_1, 1, 0), (t_2, 0, 1), (t_{i+3}, 0, 0)], \qquad i = 1, \ldots, n-2,$$

$$\delta_{n-1} = [(t_1, 1, 0), (t_2, 0, 1), (t_{n+1}, 0, 0), (t_{n+2}, 1, 0)],$$

$$\delta_n = [(t_2, 0, 1), (t_{n+1}, 0, 0), (t_{n+3}, 1, 0), (t_{n+3}, 0, 1)].$$

Then $\xi_i = 0$, $i = 1, 2, \ldots, n-2$, $\xi_{n-1} = \frac{1}{2}$, $\xi_n = 1$, and it is easy to see that for $g(x) = x^2$, $V_\Delta(g; \cdot) \nrightarrow g$ on $[0, 1]$ as $\max \Delta t_i \to 0$.

5. Two-Dimensional Marsden's Identity

For $k > 2$ let Ω denote a polygon in $\mathbb{R}^2$ and Δ_{k-2} the standard $(k-2)$-simplex in $\mathbb{R}^{k-2}$ with vertices $e^0,\ldots,e^{k-2}$ as in (2.3). Let $\Delta = \{\delta_i\}_{i=1}^n$ be a triangulation of $\Omega \times \Delta_{k-2} \subset \mathbb{R}^k$ such that each vertex lies on one of the faces $\mathscr{F}_j = \{(x, e^j)\colon x \in \Omega\}$, $j = 0,\ldots, k-2$. For $i = 1,\ldots, n$ we write $\delta_i = [v^{i0},\ldots, v^{ik}]$ as before and denote the projection of v^{ij} on Ω by $x^{ij} = (x_1^{ij}, x_2^{ij})$.

Now for any simplex δ_i, each face $\mathscr{F}_j$, $j = 0,\ldots, k-2$, contains at least one of the vertices of δ_i and hence there are two possibilities.

(i) There is one face containing three vertices of δ_i. In this case we denote these vertices by v^{i0}, v^{i1}, v^{i2}.

(ii) There are two faces, each containing exactly two vertices of δ_i. In this case we denote those on one face by v^{i0}, v^{i1} and those on the other face by v^{i2}, v^{i3}.

Now for $i = 1,\ldots, n$ we define a normalised B-spline of degree $k-2$ on Ω by

$$N_i := (k-2)!\, M_{\delta_i}. \tag{5.1}$$

By a calculation similar to that for (2.5), we find the following.

In case (i), $\operatorname{vol}_k \delta_i = |\det(x^{i1} - x^{i0}\ x^{i2} - x^{i0})|/k!$.
In case (ii), $\operatorname{vol}_k \delta_i = |\det(x^{i1} - x^{i0}\ x^{i3} - x^{i2})|/k!$.

Thus by (5.1) and (2.2) we have the following.

In case (i),

$$N_i = \frac{1}{k(k-1)} |\det(x^{i1} - x^{i0}\ x^{i2} - x^{i0})|\, M(\cdot \mid x^{i0},\ldots, x^{ik}). \tag{5.2}$$

In case (ii),

$$N_i = \frac{1}{k(k-1)} |\det(x^{i1} - x^{i0}\ x^{i3} - x^{i2})|\, M(\cdot \mid x^{i0},\ldots, x^{ik}). \tag{5.3}$$

Now for $i = 1,\ldots, n$, $j = 3,\ldots, k$, we define z^{ij} in $\mathbb{R}^2$ as follows. If case (ii) holds for δ_i, then z^{i3} is the point of intersection of the line $x^{i0}x^{i1}$ and the line $x^{i2}x^{i3}$. (These lines cannot be parallel or δ_i would be degenerate.) In all other cases $z^{ij} = x^{ij}$. Then we have the following two-dimensional version of Marsden's identity.

THEOREM 5.1. *For any $y \in \mathbb{R}^2$ and $x \in \Omega$,*

$$(y_1 y_2 - y_2 x_1 - y_1 x_2)^{k-2} = \sum_{i=1}^{n} \left\{ \prod_{j=3}^{k} (y_1 y_2 - y_2 z_1^{ij} - y_1 z_2^{ij}) \right\} N_i(x). \quad (5.4)$$

Proof. Without loss of generality we may assume $\Omega \subset \{x \in \mathbb{R}^2: x_1, x_2 \geqslant 0\}$. For $y_1, y \geqslant 1$ we let Δ_k^y denote the k-simplex in $\mathbb{R}^k$ with vertices $(y_1, 0, e^0)$, $(0, y_2, e^0)$, $(0, 0, e^i)$, $i = 0, \dots, k-2$. For large enough y_1, y_2, Δ_k^y contains $\Omega \times \{e^0\}$ and we let $P := \{(x_1, \dots, x_k) \in \Delta_k^y : (x_1, x_2) \in \Omega\}$. A simple geometric argument shows that $P = \{T_y v: v \in \Omega \times \Delta_{k-2}\}$, where for $(x, z) \in \Omega \times \Delta_{k-2}$,

$$T_y(x, z) = \left(x, \left(1 - \frac{x_1}{y_1} - \frac{x_2}{y_2}\right) z\right).$$

If δ_i^y denotes the simplex $[T_y v^{i0}, \dots, T_y v^{ik}]$, $i = 1, \dots, n$, then for large enough $y_1, y_2, \Delta^y := \{\delta_i^y\}_{i=1}^n$, is a triangulation of P.

Now for $i = 1, \dots, n$,

$$\mathrm{vol}_k\, \delta_1^y = \frac{1}{k!} \left| \det \begin{pmatrix} T_y v^{i0} & \cdots & T_y v^{ik} \\ 1 & \cdots & 1 \end{pmatrix} \right|. \quad (5.5)$$

First suppose δ_i satisfies (i). If $v^{i0} \notin \mathscr{F}_0$, let v^{il} denote the vertex of δ_i lying in $\mathscr{F}_0$. A simplification of (5.5) then gives

$$\mathrm{vol}_k\, \delta_i^y = \frac{1}{k!} \left| \det \begin{pmatrix} x^{i0} & x^{i1} & x^{i2} & x^{il} \\ \alpha_{i0} & \alpha_{i1} & \alpha_{i2} & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right| \prod_{\substack{2 \leqslant j \leqslant k \\ j \neq l}} \alpha_{ij}, \quad (5.6)$$

where $\alpha_{ij} = 1 - x_1^{ij}/y_1 - x_2^{ij}/y_2$.

By manipulating the rows of the determinant in (5.6) we get

$$\mathrm{vol}_k\, \delta_i^y = \frac{1}{k!} \left| \det \begin{pmatrix} x^{i0} & x^{i1} & x^{i2} \\ 1 & 1 & 1 \end{pmatrix} \right| \prod_{j=3}^{k} \alpha_{ij}. \quad (5.7)$$

If $v^{i0} \in \mathscr{F}_0$, then a simpler calculation also gives (5.7).

Next suppose δ_i satisfies (ii). If $v^{i0}, v^{i2} \notin \mathscr{F}_0$, then as before let v^{il} denote the vertex of δ_i lying in $\mathscr{F}_0$. A simplification of (5.5) then gives

$$\mathrm{vol}_k\, \delta_i^y = \frac{1}{k!} \left| \det \begin{pmatrix} x^{i0} & x^{i1} & x^{i2} & x^{i3} & x^{il} \\ \alpha_{i0} & \alpha_{i1} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{i2} & \alpha_{i3} & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \right| \prod_{\substack{4 \leqslant j \leqslant k \\ j \neq l}} \alpha_{ij}$$

and a manipulation of rows gives

$$\operatorname{vol}_k \delta_i^y = \frac{1}{k!} \left| \det \begin{pmatrix} x^{i0} & x^{i1} & x^{i2} & x^{i3} \\ \alpha_{i0} & \alpha_{i1} & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right| \prod_{k=4}^{k} \alpha_{ij}. \tag{5.8}$$

Now recall that z^{i3} is the point of intersection of the line $x^{i0}x^{i1}$ with the line $x^{i2}x^{i3}$ and so

$$z^{i3} = \frac{x^{i1} \det \begin{pmatrix} x^{i0} & x^{i2} & x^{i3} \\ 1 & 1 & 1 \end{pmatrix} - x^{i0} \det \begin{pmatrix} x^{i1} & x^{i2} & x^{i3} \\ 1 & 1 & 1 \end{pmatrix}}{\det(x^{i1} - x^{i0}\ x^{i3} - x^{i2})}. \tag{5.9}$$

Then expanding the determinant in (5.8) by its third row and applying (5.9) gives

$$\operatorname{vol}_k \delta_i^y = \frac{1}{k!} \det(x^{i1} - x^{i0}\ x^{i3} - x^{i2}) \left(1 - \frac{z_1^{i3}}{y_1} - \frac{z_2^{i3}}{y_2}\right) \prod_{j=4}^{k} \alpha_{ij}. \tag{5.10}$$

If $v^{i0} \in \mathscr{F}_0$ or $v^{i2} \in \mathscr{F}_0$, a simpler calculation also gives (5.10).

Now for each x in Ω, the hyperplane $\{(x_1,..., x_k)\colon (x_1, x_2) = x\}$ intersects P in the $(k-2)$-simplex σ_x with vertices $T_y(x, e^j)$, $j = 0,..., k-2$. Then

$$\begin{aligned} \frac{1}{(k-2)!}\left(1 - \frac{x_1}{y_1} - \frac{x^2}{y_2}\right)^{k-2} &= \operatorname{vol}_{k-2} \sigma_x \\ &= \sum_{i=1}^{n} M_{\delta_i^y}(x) \qquad \text{by (2.1)} \\ &= \sum_{i=1}^{n} \operatorname{vol}_k \delta_i^y M(x \mid x^{i0},..., x^{ik}) \qquad \text{for (2.2)} \\ &= \sum_{i=1}^{n} \frac{1}{(k-2)!} \left\{ \prod_{j=3}^{k} \left(1 - \frac{z_1^{ij}}{y_1} - \frac{z_2^{ij}}{y_2}\right) \right\} N_i(x) \\ &\qquad \text{by (5.2), (5.3), (5.7), (5.10)} \end{aligned}$$

which gives (5.4). ∎

COROLLARY. *For $k > 3$ and $i = 1,..., n$, let*

$$\xi_{ij} = \frac{1}{k-2} \sum_{l=3}^{k} z_j^{il}, \qquad j = 1, 2, \tag{5.11}$$

$$\xi_{ij}^{(2)} = \frac{2}{(k-2)(k-3)} \sum_{3 \leqslant l < m \leqslant k} z_j^{il} z_j^{im}, \qquad j = 1, 2, \tag{5.12}$$

$$\xi_i^{(2)} = \frac{1}{(k-2)(k-3)} \sum_{3 \leqslant l \neq m \leqslant k} z_1^{il} z_2^{im}. \tag{5.13}$$

Then for all x in Ω,

$$1 = \sum_{i=1}^{n} N_i(x), \tag{5.14}$$

$$x_j = \sum_{i=1}^{n} \xi_{ij} N_i(x), \qquad j = 1, 2, \tag{5.15}$$

$$x_j^2 = \sum_{i=1}^{n} \xi_{ij}^{(2)} N_i(x), \qquad j = 1, 2, \tag{5.16}$$

$$x_1 x_2 = \sum_{i=1}^{n} \xi_i^{(2)} N_i(x). \tag{5.17}$$

Proof. In (5.4) equate coefficients of $y_1^{k-2} y_2^{k-2}$, $y_1^{k-3} y_2^{k-2}$, $y_1^{k-2} y_2^{k-3}$, $y_1^{k-4} y_2^{k-2}$, $y_1^{k-2} y_2^{k-4}$, $y_1^{k-3} y_2^{k-3}$, respectively. ∎

6. Two-Dimensional Bernstein–Schoenberg Operators

As in Section 5 we take a polygon Ω and for $k > 2$ define a triangulation $\Delta = \{\delta_i\}_{i=1}^{n}$. For $i = 1, \ldots, n$ define N_i by (5.2), (5.3) and $\xi_i = (\xi_{i1}, \xi_{i2})$ where ξ_{i1}, ξ_{i2} are defined by (5.11). Then we define the Bernstein–Schoenberg operator V_Δ as follows.

For any function f on Ω,

$$V_\Delta(f; x) = \sum_{i=1}^{n} f(\xi_i)\, N_i(x), \qquad \forall x \in \Omega. \tag{6.1}$$

Clearly V_Δ is a positive linear operator. It follows from (5.14) and (5.15) that V_Δ reproduces polynomials of degree 1. We now assume $k > 3$ and give a two-dimensional version of Theorem 4.1.

Theorem 6.1. *Suppose that for each triangulation Δ and for each simplex $\delta_i \in \Delta$ satisfying* (ii) *of Section* 5, *the line $x^{i0} x^{i1}$ and the line $x^{i2} x^{i3}$ intersect in* supp N_i. *Then for every $f \in C(\Omega)$, $V_\Delta(f; \cdot) \to f$ uniformly on Ω if* $(1/k) \max_{1 \leq i \leq n} d_i \to 0$, *where* $d_i = \operatorname{diam}(\operatorname{supp} N_i)$.

Proof. By the Bohman–Korovkin Theorem, it is sufficient to show that $V_\Delta(f; \cdot) \to f$ uniformly on Ω for $f = g_1$, g_2 and h, where $g_1(x) = x_1^2$, $g_2(x) = x_2^2$ and $h(x) = x_1 x_2$.

Now from (6.1) and (5.16) we have for $j = 1, 2$

$$V_\Delta(g_j; x) - g_j(x) = \sum_{i=1}^{n} (\xi_{ij}^2 - \xi_{ij}^{(2)})\, N_i(x). \tag{6.2}$$

A straightforward calculation shows that for $i = 1,\dots, n$,

$$\xi_{ij}^2 - \xi_{ij}^{(2)} = \frac{1}{(k-2)^2(k-3)} \sum_{3 \leqslant l < m \leqslant k} (z_j^{im} - z_j^{il})^2$$

and thus $0 \leqslant \xi_j^2 - \xi_{ij}^{(2)} \leqslant d_i^2/2(k-2)$.

So by (6.2) and (5.14) we have

$$0 \leqslant V_\Delta(g_j; x) - g_j(x) \leqslant \frac{1}{2(k-2)} \max_{1 \leqslant i \leqslant n} d_i^2, \qquad \forall x \in \Omega.$$

Also from (6.1) and (5.17) we have

$$V_\Delta(h; x) - h(x) = \sum_{i=1}^{n} (\xi_{i1}\xi_{i2} - \xi_i^{(2)})\, N_i(x). \tag{6.3}$$

Now a straightforward calculation shows that for $i = 1,\dots, n$,

$$\xi_{i1}\xi_{i2} - \xi_i^{(2)} = \frac{1}{(k-2)^2(k-3)} \sum_{l,m=3}^{n} z_1^{il}(z_2^{il} - z_2^{im})$$

and thus $|\xi_{i1}\xi_{i2} - \xi_i^{(2)}| \leqslant Kd_i/(k-2)$, where K depends only on Ω.

So by (6.3) and (5.14) we have

$$|V_\Delta(h; x) - h(x)| \leqslant \frac{K}{k-2} \max_{1 \leqslant i \leqslant n} d_i.$$

Hence $V_\Delta(h; \cdot) \to h$ uniformly on Ω if $(1/k) \max_{1 \leqslant i \leqslant n} d_i \to 0$. ∎

7. Bernstein Polynomials

Let $x^0 = (0, 0)$, $x^1 = (1, 0)$, $x^2 = (0, 1)$ and Ω be the triangle $[x^0, x^1, x^2]$. For $k > 2$ we define a triangulation

$$\Delta = \{\delta_{ij} : i, j \geqslant 0, i + j \leqslant k-2\} \qquad \text{of } \Omega \times \Delta_{k-2} \text{ by}$$

$$\begin{aligned}\delta_{ij} = [&(1, 0, e^0), (1, 0, e^1),\dots, (1, 0, e^i), (0, 1, e^i),\\ &(0, 1, e^{i+1}),\dots, (0, 1, e^{i+j}), (0, 0, e^{i+j}), (0, 0, e^{i+j+1}),\\ &\dots, (0, 0, e^{k-2})].\end{aligned}$$

By (5.2) and (5.3), the normalised B-spline corresponding to the simplex δ_{ij} is given by

$$N_{ij} = \frac{1}{k(k-1)} M(\cdot \mid \underbrace{x^0,\dots, x^0}_{k-1-i-j}, \underbrace{x^1,\dots, x^1}_{i+1}, \underbrace{x^2,\dots, x^2}_{j+1}). \tag{7.1}$$

Now it follows from a formula of Micchelli (Corollary 2 of [12]) that

$$M(\cdot \mid \underbrace{x^0,\dots,x^0}_{k-1-i-j}, \underbrace{x^1,\dots,x^1}_{i+1}, \underbrace{x^2,\dots,x^2}_{j+1})$$

$$= \frac{k!}{(k-2-i-j)!\,i!\,j!} x_1^i x_2^j (1-x_1-x_2)^{k-2-i-j}. \tag{7.2}$$

So by (7.1), (7.2) and (6.1), the Bernstein–Schoenberg operator is given by

$$V_\Delta(f;x) = \sum_{\substack{i,j\geqslant 0\\ i+j\leqslant k-2}} f\left(\frac{i}{k-2}, \frac{j}{k-2}\right)\binom{k-2}{i}\binom{k-2-i}{j}$$

$$\times x_1^i x_2^j (1-x_1-x_2)^{k-2-i-j}, \tag{7.3}$$

and therefore comprises Bernstein polynomials (see [8]).

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